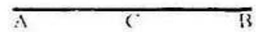
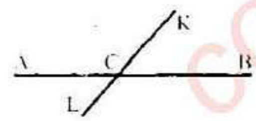
UNIT 12:
LINE BISECTORS & ANGLE BISECTORS

Definition:

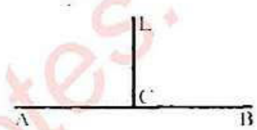
a) If C is a point on $\overline{AB}$ such that $m\overline{AC} = m\overline{BC}$, then C is said to be the midpoints of $\overline{AB}$.



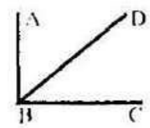
b) Any line through C (midpoint of $\overline{AB}$) is called the bisector of $\overline{AB}$ i.e. $\overleftrightarrow{KL}$ is the bisector of $\overline{AB}$.



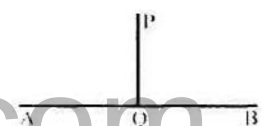
c) If a bisector of a line segment is perpendicular to the line segment as well, then it is said to be 'right-bisector'. In the figure $\overleftrightarrow{LC}$ is the right bisector of $\overline{AB}$ because it passes through its midpoint C and is also perpendicular to $\overline{AB}$.



d) If a point D exists in the interior of an angle ABC, such that $m\angle ABD = m\angle DBC$ then $\overrightarrow{BD}$ is called the bisector of angle ABC.



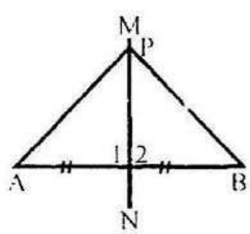
e) Distance of a point from a line is the length of perpendicular drawn from the point to the line. Distance of P from $\overline{AB}$ is $m\overline{PQ}$, where $\overline{PQ} \perp \overline{AB}$.



THEOREM 12.1

Any point on the right bisector of a line segment is equidistant from end points of the segment.

Given: A line MN intersects the line segment $\overline{AB}$ at the point R such that MN is the right bisector of $\overline{AB}$ which cuts $\overline{AB}$ at R. Let P be a point on MN.



Construction:
Join P to points A and B.

To Prove: P is equidistant from A and B i.e. $m\overline{PA} = m\overline{PB}$

Proof:

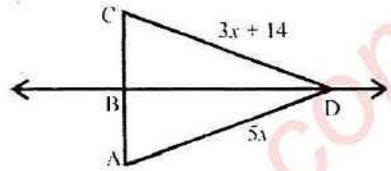
Statements	Reasons
In $\Delta PRA \longleftrightarrow \Delta PRB$	
$\overline{PR} \cong \overline{PR}$	Common
$\overline{AR} = \overline{BR}$	Given
$\angle 1 \cong \angle 2$	Right angles
$\therefore$ $\Delta PRA \cong \Delta PRB$	(S.A.S $\cong$ S.A.S)
Hence $\overline{PA} \cong \overline{PB}$	Corresponding sides of congruent triangles
This means that P is equidistant from A and B.	

EXAMPLE

$\overleftrightarrow{BD}$ **is the perpendicular bisector of** $\overline{AC}$. **Find** AD.

Solution:

$AD = CD$	perpendicular bisector theorem
$5x = 3x + 14$	substitute.
$x = 7$	solve for x
$AD = 5x$	
$= 5(7)$	
$= 35$	



THEOREM 12.2

Any point equidistant from the end points of a line segment is on the right bisector of it.

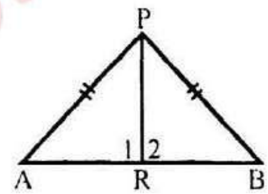
Given: P is equidistant from the end points of $\overline{AB}$ i.e. $\overline{PA} \cong \overline{PB}$

To Prove:

P lies on the right bisector of $\overline{AB}$.

Construction:

Draw a perpendicular on $\overline{AB}$, meeting $\overline{AB}$ at R.



Proof:

Statements	Reasons
In $\Delta PRA \longleftrightarrow \Delta PRB$	
$\overline{PA} \cong \overline{PB}$	Given
$\overline{PR} \cong \overline{PR}$	Common
$m\angle 1 \cong m\angle 2 = 90^\circ$	Construction
$\therefore$ $\Delta PRA \cong \Delta PRB$	(S.A.S $\cong$ S.A.S)
Hence $\overline{PR} \cong \overline{PR}$	Corresponding sides of congruent triangle
$\therefore$ $\overline{PR}$ is the right bisector of $\overline{AB}$ or	
P lies on the right bisector of $\overline{PR}$	

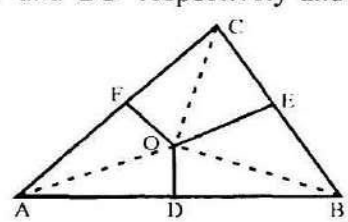
THEOREM 12.3

Statement: The right bisectors of the sides of a triangle are concurrent.

Given: In ΔABC, $\overline{DO}$ and $\overline{EO}$ are the right bisectors of $\overline{AB}$ and $\overline{BC}$ respectively and intersects each other at O. Join O to F which is midpoint of $\overline{AC}$.

To Prove: The right bisectors of the sides $\overline{AB}$, $\overline{BC}$, $\overline{AC}$ of a triangle are concurrent at O.

Construction: Draw the right bisectors of $\overline{AB}$ and $\overline{BC}$ which intersect at O. Join O to A, B, C.



Proof:

Statements	Reasons
In ΔAOB	

Statement	Reasons
O lies on the right bisector of $\overline{AB}$	Given
$\therefore \quad \overline{AO} \cong \overline{BO} \longrightarrow (1)$	O is on right bisector of $\overline{AB}$
Similarly in ΔBOC	
$\overline{BO} \cong \overline{CO} \longrightarrow (2)$	
Hence $\overline{AO} \cong \overline{CO}$	From (1) and (2)
Or O is on the right bisector of $\overline{AC}$. Hence the right bisectors of three sides of a triangle are concurrent at point O.	O is equidistant from A and C

THEOREM 12.4

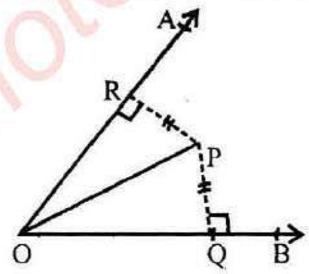
Statement: Any point on the bisector of an angle is equidistant from its arms.

Given: $\overrightarrow{OM}$ is the bisector of $\angle AOB$.

P is any point on $\overrightarrow{OM}$.

To Prove:

P is equidistant from the arms of $\angle AOB$ i.e. $\overline{PR} \cong \overline{PQ}$



Construction:

Draw $\overline{PR} \perp \overline{OA}$ and $\overline{PQ} \perp \overline{OB}$.

Proof:

Statement	Reasons
In $\quad \Delta PRO \longleftrightarrow \Delta PQO$	
$\angle POR \cong \angle POQ$	Given
$\angle PQO \cong \angle PRO$	Right angles
$\overline{PO} \cong \overline{PO}$	Common
$\therefore \quad \Delta PRO \cong \Delta PQO$	(A.A.S $\cong$ A.A.S)
Hence $\overline{PR} \cong \overline{PQ}$	Corresponding sides of congruent triangles
i.e. P is equidistant from $\overrightarrow{BA}$ and $\overrightarrow{BC}$.	

THEOREM 12.5

Statement: Any point inside an angle, equidistant from its arms, is on its bisector.

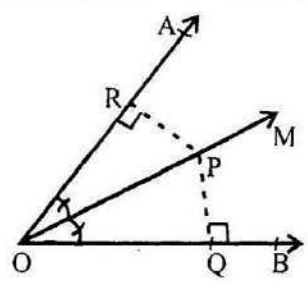
Given: P is a point lies inside $\angle AOB$,

Such that $\overline{PR} \cong \overline{PQ}$ where $\overline{PR} \perp \overline{OA}$ and $\overline{PQ} \perp \overline{OB}$

To Prove: $\overrightarrow{OP}$ lies on the bisector of $\angle AOB$

Construction:

Joint P to O. Draw perpendicular $\overline{PR}$ and $\overline{PQ}$ on $\overrightarrow{OA}$ and $\overrightarrow{OB}$.



Proof:

Statements	Reasons
In $\Delta OPR \longleftrightarrow \Delta OPQ$	
$\overline{PO} \cong \overline{PO}$	Common
$\overline{PR} \cong \overline{PQ}$	Given
$\angle PRO \cong \angle PQO$	Right angles
$\therefore$ $\Delta OPR \cong \Delta OPQ$	(S.A.S $\cong$ S.A.S)
Hence $\angle POR \cong \angle POQ$	Corresponding angles of the congruent triangle
Or $\overline{OP}$ is the bisector of $\angle AOB$	

EXAMPLE

For what values of x does P lies on the bisector of $\angle A$?

Solution:

From the converse of the angle bisector theorem P lies on the bisector of $\angle A$ if P is equidistant from the sides of $\angle A$, so when $BP = CP$.

$BP = CP$	set segment lengths equal
$x + 3 = 2x - 1$	substitute expressions for segments lengths
$4 = x$	solve for x

Thus the point P lies on the bisector of $\angle A$ when $x = 4$.

An angle bisector of a triangle is the bisector of the interior angle of the triangle.

THEOREM 12.6

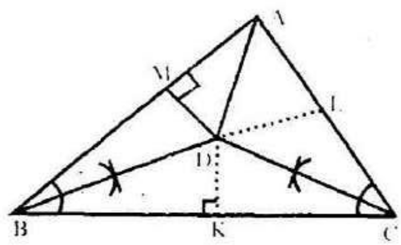
Statement: The bisectors of the angles of a triangle are concurrent.

Given: $\overline{BD}$ and $\overline{CD}$ are the bisectors of $\angle B$ And $\angle C$ of ΔABC, which intersects each other at D. Join D with A.

To Prove: $\overline{DA}$ is bisector of $\angle A$.

Construction:

From D, draw $\overline{DK} \perp \overline{BC}$, $\overline{DL} \perp \overline{CA}$ and $\overline{DM} \perp \overline{AB}$.

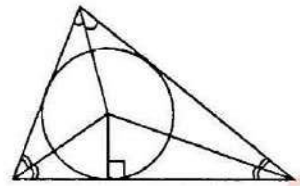


Proof:

Statements	Reasons
As D lies on bisector of $\angle B$	Given
$\therefore$ $\overline{DK} \cong \overline{DM} \longrightarrow (1)$	Distance of D from the arm of $\angle B$
Similarly D lies on the bisector of $\angle C$	
$\therefore$ $\overline{DK} \cong \overline{DL} \longrightarrow (2)$	Distance of D from the arms of $\angle C$
Hence $\overline{DL} \cong \overline{DM}$	From (1) and (2)
i.e. D lies on the bisector of $\angle A$	D is equidistant from L and M
Or $\overline{AD}$ is the bisector of $\angle A$	
Or the bisectors of the angles of the ΔABC are concurrent.	

Definition:

The point of concurrency of the three angle bisectors of a triangle is called the "Incentre" of the triangle. The "Incentre" always lies inside of the triangle. The "Incentre" is equidistant from the three sides of the triangle.



EXERCISE 12.1

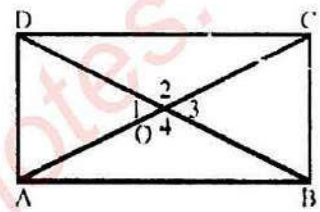
Q1: If the diagonals of a quadrilateral are the right bisectors of each other, then prove that all the sides of the quadrilateral are congruent.

Given: ABCD is a quadrilateral. $\overline{AC}$

And $\overline{BD}$ are the two diagonals meet at O

Such that $\overline{OA} \cong \overline{OC} \cong \overline{OB} \cong \overline{OD}$

$m\angle 1 \cong m\angle 2 \cong m\angle 3 \cong m\angle 4$



To Prove: $\overline{AB} \cong \overline{BC} \cong \overline{CD} \cong \overline{AD}$

Proof:

Statements	Reasons
In $\Delta BOC \longleftrightarrow \Delta DOC$	
$\overline{OC} \cong \overline{OC}$	Common
$\overline{OB} \cong \overline{OD}$	Given
$m\angle 2 \cong m\angle 3$	Given
$\therefore$ $\Delta BOC \cong \Delta DOC$	(S.A.S $\cong$ S.A.S)
So $\overline{BC} \cong \overline{CD}$	Corresponding sides of congruent triangles
Similarly we can prove that $\overline{CD} \cong \overline{DA}$ and $\overline{DA} \cong \overline{AB}$	

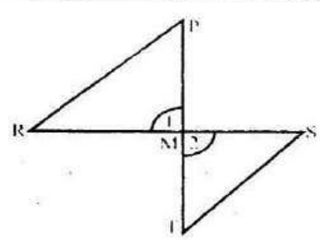
Q2: If $\overline{PR}$ and $\overline{TS}$ are $\perp$ $\overline{RS}$, $\overline{PM} \cong \overline{MT}$ and $m\angle PMT = 90^\circ$. Prove that $\Delta PRM \cong \Delta MTS$.

Given:

$\overline{PM} \cong \overline{MT}$

$m\angle 1 \cong m\angle 2 = 90^\circ$

$\overline{RM} \cong \overline{MS}$



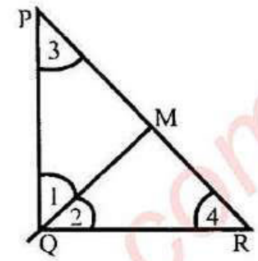
To Prove:

$\Delta PRM \cong \Delta MTS$

Proof:

Statements	Reasons
In $\Delta PRM \longleftrightarrow \Delta MTS$	
$\overline{PM} \cong \overline{MT}$	Given
$m\angle 1 \cong m\angle 2$	Each is 90° (Given)
$\overline{RM} \cong \overline{MS}$	Given
$\therefore$ $\Delta PRM \cong \Delta MTS$	S.A.S $\cong$ S.A.S

Q3: If $\angle 3 \cong \angle 4$ and $\overline{QM}$ bisectors $\angle PQR$, prove that M is the midpoint of $\overline{PR}$.



Given:

$\angle 3 \cong \angle 4$, QM bisectors $\angle PQR$

Such that $\angle 1 \cong \angle 2$

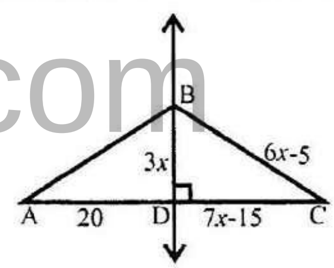
To Prove:

$\overline{PM} \cong \overline{MR}$ or M is the midpoint of $\overline{PR}$.

Proof:

Statements	Reasons
In $\Delta PMQ \longleftrightarrow \Delta MQR$	
$m\angle 1 \cong m\angle 2$	Given
$m\angle 3 \cong m\angle 4$	Given
$\overline{MQ} \cong \overline{MQ}$	Common
$\therefore$ $\Delta PMQ \cong \Delta MQR$	(A.A.S $\cong$ A.A.S)
Hence $\overline{PM} \cong \overline{MR}$	Corresponding sides

Q4: Find the length of AB.



Solution:

From the figure, $\Delta ABD \cong \Delta BDC$ and both are right triangles.

So $\overline{AB} \cong \overline{BC}$ = hypotenuse

Now $(AB)^2 = (BC)^2 \Rightarrow (5x)^2 = (4x+3)^2$

$25x^2 = 16x^2 + 24x + 9$

OR $25x^2 - 16x^2 - 24x - 9 = 0$ $\quad\Rightarrow 9x^2 - 24x - 9 = 0$

$3(3x^2 - 8x - 3) = 0$ $\quad\Rightarrow 3x^2 - 8x - 3 = 0$

$\Rightarrow 3x^2 - 9x + x - 3 = 0$ $\quad\Rightarrow 3x(x-3) + 1(x-3) = 0$

$(x-3)(3x+1) = 0$

$\therefore x - 3 = 0 \Rightarrow x = 3$

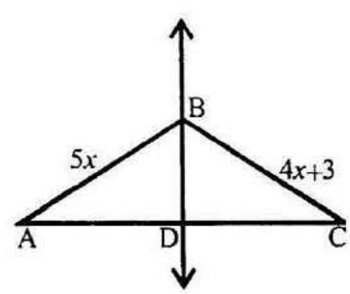
Now length of $\overline{AB} = 5x = 5(3) = 15$ Ans.

Q5: In the diagram, $\overleftrightarrow{BD}$ is the perpendicular bisector of $\overline{AC}$.

i) What segment lengths are equal?

ii) What is the value of x?

iii) Find AB.



Solution:

In $\Delta ABD \longleftrightarrow \Delta BDC$

$\overline{AD} = \overline{DC}$

$\because$ $\overline{BD}$ is perpendicular bisector of $\overline{AC}$

So $\overline{AB} = \overline{BC}$ (Each is hypotenuse of same triangles)

Now $\overline{AD} = \overline{DC} \Rightarrow 7x - 15 = 20$

$\Rightarrow 7x = 20 + 15 = 35$

$\Rightarrow x = \dfrac{35}{7} = 5$

i) Here $\overline{AB} = \overline{BC}$ are equal segment

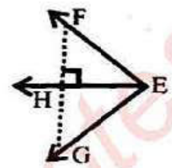
ii) $x = 5$

iii) $\overline{AB} = 6x - 5 = 6(5) - 5 = 30 - 5 = 25$ Ans.

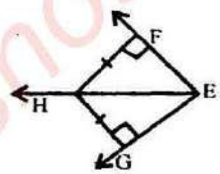
Q6: Can we conclude that $\overleftrightarrow{EH}$ bisects $\angle FEG$?

Solution:

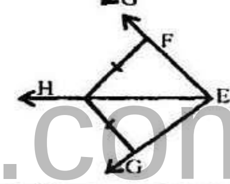
i) No, we cannot conclude that $\overline{EH}$ is the bisector of $\angle FEG$ because $\angle FHE = 90^o$ but $\angle GHE \neq 90^o$



ii) Yes, here we can conclude that $\overline{EH}$ bisects $\angle FEG$ because $\overline{FH} \cong \overline{GH}$ and $\angle F = \angle G = 90^o$



iii) No, here again $\overline{EH}$ is not the bisector of $\angle FEG$.



Q7: Find the values of x:

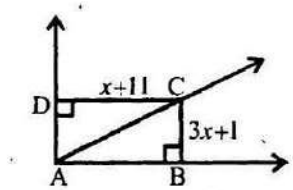
Solution:

i) Since $\angle B = \angle D = 90^o$

So $\overline{DC} = \overline{BC} \Rightarrow 3x + 1 = x + 11$

$\Rightarrow 3x - x = 11 - 1$

$\Rightarrow 2x = 10 \Rightarrow \boxed{x = 5}$



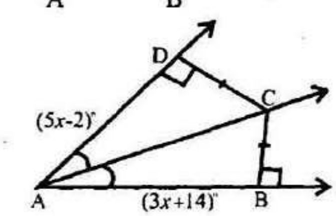
ii) Since $\overline{AC}$ is the angle bisector of $\angle BAD$

So $\angle BAC = \angle CAD$

$\Rightarrow 3x + 14 = 5x - 2$

$\Rightarrow 3x - 5x = -2 - 14$

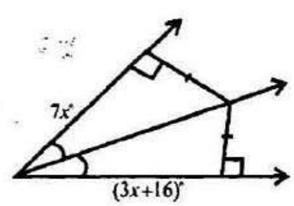
$\cancel{-}2x = \cancel{-}16 \Rightarrow x = \dfrac{16}{2} \Rightarrow x = 8$



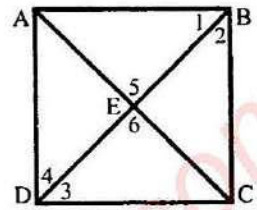
iii) Here $7x = 3x + 16$

$7x - 3x = 16$

$4x = 16 \Rightarrow x = \dfrac{16}{4} = 2$



Q8: Prove that the diagonals of a square are the right bisectors of each other.



Given:

ABCD is a square

in which $\overline{AC}$ and $\overline{BD}$ are diagonals.

To Prove:

$\overline{AC}$ and $\overline{BD}$ are the right bisectors.

Proof:

Statement	Reasons
In ΔABD, $\angle 1 \cong \angle 4$	Angle opposite congruent sides
$\angle 2 \cong \angle 4$	Alternate angles
$\therefore$ $\angle 1 \cong \angle 2$	
In $\Delta AEB \longleftrightarrow \Delta BEC$	
$\overline{AB} \cong \overline{BC}$	All sides of square are equal
$\overline{BE} \cong \overline{BE}$	Common
$\angle 1 \cong \angle 2$	Proved
$\therefore$ $\Delta AEB \cong \Delta BEC$	(S.A.S)
Hence $\overline{AE} \cong \overline{EC}$	Corresponding sides
$\therefore$ $\angle BEA + \angle BEC = 180^o$	Supplementary angles
$\angle BEA \cong \angle BEC = 90^o$	
Hence $\overline{AB} \perp \overline{BD}$ and $\overline{CE} \perp \overline{BD}$.	

REVIEW EXERCISE 12

Q1: Select the correct answer and write the corresponding letter a, b, c or d in the box.

i) Which of the following are concurrent?

(a) Angles bisectors of a triangle
(b) Perpendicular bisectors of the sides of a triangle
(c) Medians of a triangle ✓ **(d) All of these**

ii) Which of the following sometimes is inside a triangle, sometimes coincides with a side of a triangle, and sometimes falls outside of a triangle?

(a) The base ✓ **(b) The altitude**
(c) The median (d) The angle bisector

iii) Perpendicular bisectors of a triangle are:

(a) Congruent ✓ **(b) Concurrent**
(c) Parallel to each other (d) Perpendicular to each other

iv) In which triangle the perpendicular bisector of the base passes through its vertex angle?

(a) Right angled (b) Scalene
✓ **(c) Isosceles** (d) Acute-angled

v) In $\triangle ABC$, medians AD, BE, and CF intersect at G. If $CF = 24$, what is the length of FG?

✓ **(a) 8** (b) 12 (c) 10 (d) 16

vi) The angle bisectors of a triangle meet at a point which is equidistant from............ of the triangle.

(a) The vertices ✓ **(b) The sides**
(c) Midpoints of the sides (d) All of these

vii) In an equilateral triangle, all the perpendicular bisector are:

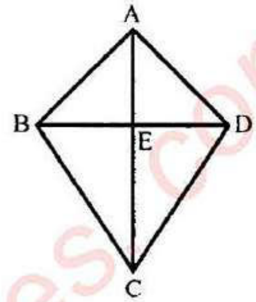
(a) Congruent ✓ **(b) Concurrent**
(c) The angle bisector as well (d) Parallel

viii) Point of intersection of the angle bisectors of a triangle is equidistant from........... of the triangle.

(a) The vertices ✓ **(b) The sides**
(c) Midpoints of the sides (d) All of the above

Q2: Prove that if both pairs of opposite sides of a quadrilateral are congruent, then the in the given figure $\overline{AB} \cong \overline{AD}$ and $\overline{BC} \cong \overline{DC}$. Prove that $\overline{AC} \perp \overline{BD}$ and $\overline{BE} \cong \overline{DE}$. y are also parallel.

Solution:



Given $\overline{AB} \cong \overline{AD}$

And $\overline{BC} \cong \overline{DC}$

To Prove:

i) $\overline{BE} \cong \overline{DE}$

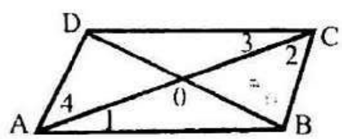
ii) $\overline{AC} \perp \overline{BD}$

Proof:

Statement	Reasons
In $\Delta ABE \longleftrightarrow \Delta ADE$	
$\overline{AB} \cong \overline{AD}$	Given
$\overline{AE} \cong \overline{AE}$	Common
$m\angle AEB \cong m\angle AED$	Opposite angles of $\overline{AB}$ and $\overline{AD}$
$\therefore$ $\Delta ABE \cong \Delta ADE$	S.A.S
Hence $\overline{BE} \cong \overline{DE}$	Corresponding sides of congruent triangles
But $m\angle AEB \cong m\angle AED$	Corresponding sides of congruent triangles
$m\angle 3 \cong m\angle 4$	
But $m\angle 3 + m\angle 4 = 180^o$	Supplementary angles
Hence $m\angle 3 = m\angle 4 = 90^o$	
$\therefore$ $\overline{AC}$ is perpendicular to $\overline{BD}$	

Q3: Prove that the diagonals of a rhombus are the right bisectors of each other.

Solution:



Given $\overline{AC}$ and $\overline{BC}$ are the diagonals of rhombus ABCD.

To Prove: $\overline{AC}$ and $\overline{BD}$ bisect each other at right angles.

Proof:

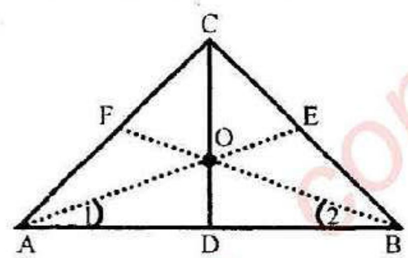
Statement	Reasons
In $\Delta ABC \longleftrightarrow \Delta BAD$	
$\overline{AB} \cong \overline{AB}$	Common
$m\angle ABC \cong \angle BAD$	Each is of opposite side of rhombus
$BC \cong AD$	S.A.S
$\Delta ABC \cong \Delta BAD$	
$\overline{AC} \cong \overline{BD}$	Corresponding sides of congruent triangles
$\overline{AB} \cong \overline{BC}$	

Q4: Prove that bisectors of the base angles of an isosceles triangle intersect each other at the right bisector of the base.

Solution:

Given in ΔABC,

$m\angle A \cong m\angle B$



To Prove:

$\overline{OD}$ is the right bisector of $\overline{AB}$ or $\overline{OD} \perp \overline{AB}$.

Proof:

Statement	Reasons
In $\Delta AOD \longleftrightarrow \Delta BOD$	O is point of intersection of the bisectors of $\angle A$ and $\angle B$.
$\overline{AO} \cong \overline{BO}$	
$m\angle 1 \cong m\angle 2$	Bisectors of two congruent angles
$\therefore$ $\Delta AOD \cong \Delta BOD$	S.A.S
Hence $m\angle 3 \cong m\angle 4$	Corresponding angles
But $m\angle 3 + m\angle 4 = 180^{o}$	Supplementary angles
$\Rightarrow$ $m\angle 3 = 90^{o}$	Hence $\overline{OD} \perp \overline{AB}$

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Additional MCQs of Unit 12:
Line Bisectors and Angle Bisectors

1. Any point on..........of a lien segment is equidistant from its end points..........
(a) Angle bisector (b) Median (c) Right bisector (d) none
✓ **Ans. (c) Right bisector**

2. The right bisectors of the sides of a triangle are..........
(a) Parallel (b) Concurrent (c) Perpendicular (d) none
✓ **Ans. (b) Concurrent**

3. The bisectors of the..........of a triangle are concurrent.
(a) Angles (b) Sides (c) Vertices (d) none
✓ **Ans. (a) Angles**

4. Bisector is a line that divides a segment into..........congruent parts.
(a) One (b) Two (c) Three (d) Four
✓ **Ans. (b) Two**

5. The point where a bisector intersects a segment is the..........of the segment.
(a) End point (b) Center point (c) Midpoint (d) none
✓ **Ans. (c) Midpoint**

6. A bisector which is perpendicular to a segment is called..........
(a) Median (b) Right bisector (c) Altitude (d) none
✓ **Ans. (a) Right bisector**

7. An angle bisector is a line which divides..........into two congruent parts.
(a) Sides (b) Triangles (c) An angle (d) none
✓ **Ans. (c) An angle**

8. A line or ray which is perpendicular to the side of a triangle is called..........
(a) Altitude (b) Bisector (c) Median (d) none
✓ **Ans. (a) Altitude**

9. The point of concurrency of the three angle bisectors of a triangle is called.........
(a) Centroide (b) The incentre (c) Bisector (d) none
✓ **Ans. (b) The incentre**

10. The incentre is..........from the three sides of the triangle.
(a) Near (b) At different distance (c) Equidistant (d) none
✓ **Ans. (c) At different distance**